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Star algebra spectroscopy

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Abstract: The spectrum of the infinite dimensional Neumann matrices M^{11}, M^{12} and M^{21} in the oscillator construction of the three-string vertex determines key properties of the star product and of wedge and sliver states. We study the spectrum of eigenvalues and eigenvectors of these matrices using the derivation $K_1 = L_1 + L_{-1}$ of the star algebra, which defines a simple infinite matrix commuting with the Neumann matrices. By an exact calculation of the spectrum of K_1, and by consideration of an operator generating wedge states, we are able to find analytic expressions for the eigenvalues and eigenvectors of the Neumann matrices and for the spectral density. The spectrum of M^{11} is continuous in the range $[-1/3, 0)$ with degenerate twist even and twist odd eigenvectors for every eigenvalue except for $-1/3$.

Keywords: Bosonic Strings, D-branes.

Contents

1. Introduction and summary

The star algebra of open string field theory (OSFT) [1] is an infinite dimensional associative algebra on a space of open string fields. While a precise and abstract mathematical characterization of this algebra is not yet available — mostly because it seems unclear how to restrict the space of open string fields to a suitable subspace where the desired axioms hold — a description of the star product in terms of oscillator expansions, or in terms of conformal field theory correlators, affords a concrete operational definition that can be used to star multiply certain string fields unambiguously.

Indeed, shortly after the construction of OSFT, explicit oscillator representations of the star product in terms of a three string vertex became available and explicit tests of the axiomatic properties and of the formulation were done [2]–[5]. This construction requires the choice of a specific conformal field theory (CFT), and the most familiar one corresponds to the background of a space-filling D25 brane. In this case the matter part of the CFT is that of 26 free bosons, and the ghost part of the CFT is that of the (b, c) system. In the matter part of the oscillator construction, the three string vertex is built as an exponential of a quadratic form in the matter oscillators. The oscillators have mode labels extending over an infinite range, and string state space labels, extending over three values. The matrices V^{rs}_{mn} defining these quadratic forms, with $r, s = 1, 2, 3$, and $0 \leq m, n \leq \infty$ go under the name of Neumann coefficients and they encode the concrete definition of star multiplication. In many cases it is convenient to treat the zero modes separately and regard V^{rs}_{mn} for fixed r, s as an infinite dimensional matrix with indices $m, n \geq 1$. We thus have nine infinite matrices. It turns out that out of these nine matrices, cyclicity and symmetry properties imply that the information is contained in three matrices $M^{11} = CV^{11}$, $M^{12} = CV^{12}$ and $M^{21} = CV^{21}$, where C is the twist matrix $C_{mn} = (-1)^m \delta_{mn}$. These matrices formally commute and as we will see they share eigenvectors. There are additional relations which for a given eigenvector allow us to relate the eigenvalues of M^{12} and M^{21} to those of M^{11}. Therefore the study of the spectral properties of M^{11} suffices. For brevity we will simply call $M \equiv M^{11}$.

It has become clear over the last year that the spectrum of M controls several important properties of star products. For example, the normalization of star algebra projectors such as the sliver state, requires in the matter sector determinant factors involving the matrix M and the divergences in such factors are controlled by the spectrum of M [6, 7]. Similarly, in vacuum string field theory [8, 9, 10], the algebraic prediction of ratios of tensions for D-branes of different dimensionalities involves ratios of determinants of M and an analogous matrix that includes oscillators with zero mode numbers [7]. Finally, formal properties, such as the commutation of the various M matrices can be rendered anomalous in the presence of inverses of the factor $(1 + 3M)$ because of the presence of an eigenvalue $\mu = -1/3$ of M [11]–[15]. Such manipulations are required in testing proposals for tachyon fluctuations in vacuum string field theory [11, 12, 15].

The matrix M not only describes the essence of the three string vertex but is also intimately related to the so-called wedge states [16], and to the sliver state. Indeed, if we star multiply two vacuum states we get a wedge state whose Neumann matrix is precisely M. All matrices defining wedge states commute with M and in fact have simple expressions in terms of M. This is also the case for the sliver. Thus knowledge of the spectral properties of M allows us to understand wedge states quite completely.

In this paper we carry out a complete analysis of the spectrum of eigenvalues of M and also find the corresponding eigenvectors. We begin by introducing the various conventions and definitions in section 2. In section 3 we establish the existence of an eigenvector of M with eigenvalue $-1/3$ that makes the matrix $(1 + 3M)$ singular. In section 4 we introduce a new matrix K_1 with a continuous non-degenerate spectrum and find all its eigenvalues and eigenvectors analytically. This matrix K_1 is defined as the action of the star algebra derivation $K_1 = L_1 + L_{-1}$ on the space of positively moded oscillators. In section 5 we show that the matrix K_1 commutes with M, M^{12} and M^{21}. This together with the non-degeneracy of the K_1 spectrum implies that all the eigenvectors of K_1 are eigenvectors of M, M^{12} and M^{21}. We also find the precise relation between the eigenvalues of K_1 and M, and give a functional interpretation of the eigenvalue equations. This interpretation extends the observation of Moore and Taylor [14] that the C-odd eigenvector of M with eigenvalue $-1/3$ implies a flat direction in the sliver functional.

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While it is in principle possible the M has eigenvectors that are not eigenvectors of K_1 — for example, a C even eigenvector of eigenvalue $(-1/3)$ — our numerical experiments suggest that we are not missing any piece of the spectrum. We thus believe that the continuous spectrum of K_1 exhausts the continuous spectrum of M. The issue of the C even eigenvector is subtle since it is a vector that would be included in addition to the continuous spectrum, and thus level expansion experiments do not provide much insight. Our analysis could not establish that this eigenvector belongs to the spectrum of M and we believe that it does not. Similar remarks apply to the matrices M^{12} and M^{21}.

Section 6 is devoted to the study of the spectral density of K_1 and M. For this purpose we consider the approximation of these matrices by $L \times L$ matrices. We find an explicit analytic expression for the density of eigenvectors in the large L limit and compare it with numerical results finding reasonable agreement. We conclude in section 7 with some open questions and remarks.

Brief summary of results. Since the analysis of the paper is somewhat technical, we shall summarize the main results here. The $*$-algebra derivation K_1 is represented on the space of positively moded oscillators by a symmetric matrix K_1 (equation (4.8)). The spectrum of K_1

$$\mathrm{K}_1 v^{(\kappa)} = \kappa v^{(\kappa)}\,, \tag{1.1}$$

exists for κ a real continuous parameter in the range $-\infty < \kappa < \infty$ and is nondegenerate. For each κ, the eigenvector $v^{(\kappa)}$ has components $v_n^{(\kappa)}$, with $n \geq 1$ given by the relation:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\, v_n^{(\kappa)} z^n = \frac{1}{\kappa}\Big(1 - \exp(-\kappa \tan^{-1} z)\Big)\,. \tag{1.2}$$

The derivation property of K_1 ensures that $[\mathrm{K}_1, M] = [\mathrm{K}_1, M^{12}] = [\mathrm{K}_1, M^{21}] = 0$. This together with the non-degeneracy of the K_1 spectrum implies that the eigenvectors of K_1 are eigenvectors of M, M^{12} and M^{21}. If we denote by $\mu(\kappa), \mu^{12}(\kappa)$ and $\mu^{21}(\kappa)$ the eigenvalues associated to $v^{(\kappa)}$ for M, M^{12}, and M^{21} respectively, we find that

$$\begin{aligned}
\mu(\kappa) &= -\frac{1}{1 + 2\cosh(\pi\kappa/2)}\,,\\
\mu^{12}(\kappa) &= -(1 + \exp(\pi\kappa/2))\,\mu(\kappa)\,,\\
\mu^{21}(\kappa) &= -(1 + \exp(-\pi\kappa/2))\,\mu(\kappa)\,.
\end{aligned} \tag{1.3}$$

Note that $\pm\kappa$ give the same value of μ, thus each eigenvalue of M, except for $\mu(0) = -1/3$, is doubly degenerate. Thus the spectrum of M lies on the interval $[-1/3, 0)$ and is doubly degenerate except at $-1/3$.[1] It also follows from the above that as $\kappa \in (-\infty, \infty)$, μ^{12} grows monotonically from zero to one, while μ^{21} decreases monotonically from one to zero. Thus, both M^{12} and M^{21} have non-degenerate spectra in the interval $(0, 1)$.

[1] As mentioned before, a C-even candidate at $-1/3$ exists, but it is not clear it properly belongs to the spectrum of M.

The eigenvectors $v^{(\kappa)}$ and $v^{(-\kappa)}$ are exchanged under the twist transformation. The degeneracy of M allows us to introduce twist eigenstates that are also M eigenstates:

$$v_{\pm}^{(\kappa)} = \frac{1}{2}(v_n^{(-\kappa)} \mp v_n^{(\kappa)}) . \tag{1.4}$$

For $\kappa = 0$ we have a single C-odd eigenvector of M with eigenvalue $\mu(0) = -1/3$. The eigenvector is defined by taking the right hand side of (1.2) to be simply $\tan^{-1}(z)$.

If we approximate K_1 by a matrix of size $L \times L$, the eigenvalues κ of K_1 become discrete. For large L the eigenvalues approach a uniform distribution with density

$$\rho_{\mathrm{K}_1}^L(\kappa) = \frac{1}{2\pi}\ln L , \tag{1.5}$$

where $\int_{\kappa_1}^{\kappa_2}\rho_{\mathrm{K}_1}^L(\kappa)d\kappa$ gives the number of eigenvalues in the interval (κ_1,κ_2). With the same finite approximation of M the degeneracy between C-even and C-odd eigenvectors is lifted. Using (1.3) and (1.5) one can easily find the density of states in μ space to be

$$\rho_M^L(\mu) = \frac{2}{\pi^2}\,\frac{1}{|\mu|\sqrt{(1+3\mu)(1-\mu)}}\,\ln L . \tag{1.6}$$

2. Notation and definitions

The star product of two states $|A\rangle$ and $|B\rangle$ in the matter part of the conformal field theory is given by,[2]

$$|A *^m B\rangle_3 = \; {}_1\langle A|\; {}_2\langle B|V_3\rangle , \tag{2.1}$$

where the three string vertex $|V_3\rangle$ is given by

$$|V_3\rangle = \int d^{26}p_{(1)}\, d^{26}p_{(2)}\, d^{26}p_{(3)}\, \delta^{(26)}(p_{(1)}+p_{(2)}+p_{(3)})\, \exp(-E)\, |0,p\rangle_{123} , \tag{2.2}$$

with

$$E = \frac{1}{2}\sum_{\substack{r,s\\ m,n\ge 1}} \eta_{\mu\nu} a_m^{(r)\mu\dagger} V_{mn}^{rs} a_n^{(s)\nu\dagger} + \sum_{\substack{r,s\\ n\ge 1}} \eta_{\mu\nu} p_{(r)}^{\mu} V_{0n}^{rs} a_n^{(s)\nu\dagger} + \frac{1}{2}\sum_r \eta_{\mu\nu} p_{(r)}^{\mu} V_{00}^{rr} p_{(r)}^{\nu} . \tag{2.3}$$

Here $a_m^{(r)\mu}$, $a_m^{(r)\mu\dagger}$ are non-zero mode matter oscillators[3] acting on the r-th string state normalized so that

$$[a_m^{(r)\mu}, a_n^{(s)\nu\dagger}] = \eta^{\mu\nu}\,\delta_{mn}\,\delta^{rs} , \qquad m,n \ge 1 . \tag{2.4}$$

$p_{(r)}$ is the 26-component momentum of the r-th string, and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacuum of the three strings, annihilated by the non-zero

[2] Our convention of $*$-product is the same as that defined in ref. [17]. The V^{rs} appearing in eq. (2.3) are transpose of the corresponding matrices given in [7, appendix A]. With this the $*$-product defined here agrees with that given in ref. [2].

[3] In our notation $i\sqrt{2}\partial X^{\mu}(z) = \sqrt{2}p^{\mu} + \sum_{n\neq 0}\sqrt{|n|}a_n^{\mu}z^{-n-1} = \sum_n \alpha_n^{\mu} z^{-n-1}$, and $\partial X^{\mu}(z)\partial X^{\nu}(w) \simeq -\eta^{\mu\nu}/2(z-w)^2$.

mode annihilation operators $a_m^{(r)\mu}$, and eigenstate of the momentum operator of the rth string with eigenvalue $p_{(r)}^{\mu}$. $|p\rangle$ is normalized as

$$\langle p|p'\rangle = (2\pi)^{26}\delta^{26}(p+p')\,. \tag{2.5}$$

The coefficients V_{mn}^{rs} for $0 \leq m,n < \infty$ can be calculated by standard methods [2, 3, 4, 5].

We define by V^{rs} the matrices V_{mn}^{rs} with $m,n \geq 1$, and by C_{mn} the twist matrix $(-1)^m\delta_{mn}$. We also define:

$$M^{rs} = CV^{rs}\,. \tag{2.6}$$

Cyclic symmetry relates these matrices so that there are only three independent matrices M^{11}, M^{12} and M^{21}. These matrices commute with each other and are real symmetric. Furthermore, we have the relations:

$$M^{12} + M^{21} = 1 - M^{11}\,, \qquad M^{12}M^{21} = M^{11}(M^{11}-1) \tag{2.7}$$

which allow us to determine the eigenvalues of M^{12} and M^{21} in terms of those of M^{11}. Our main goal in this paper will be the determination of the eigenvectors and eigenvalues of the matrix M^{11}. For convenience of notation, from now on we shall denote the matrices M^{11} and V^{11} by M and V respectively.

Finally, we note that in terms of the matrices M^{rs} we can define projection operators [17]:

$$\begin{aligned}
\rho_1 &= (1+T)^{-1}(1-M)^{-1}\Big((M^{12}(1-TM) + T(M^{21})^2\Big)\,,\\
\rho_2 &= (1+T)^{-1}(1-M)^{-1}\Big((M^{21}(1-TM) + T(M^{12})^2\Big)\,,
\end{aligned} \tag{2.8}$$

where

$$T = (2M)^{-1}\Big(1 + M - \sqrt{(1+3M)(1-M)}\Big)\,. \tag{2.9}$$

ρ_1 and ρ_2 can be shown to satisfy:

$$\rho_1^2 = \rho_1\,, \qquad \rho_2^2 = \rho_2\,, \qquad \rho_1\rho_2 = 0\,, \qquad \rho_1 + \rho_2 = 1\,, \tag{2.10}$$

and

$$\rho_2 = C\rho_1 C\,. \tag{2.11}$$

We conclude this section by giving the explicit expressions for the matrix M. Following [2, 3] we have

$$\begin{aligned}
M_{mn} &= -\frac{2}{3}\frac{\sqrt{mn}}{m^2-n^2}\Big(mA_mB_n - nA_nB_m\Big)\,, && m+n = \text{ even}\,, \quad m \neq n\,,\\
M_{mn} &= 0\,, && m+n = \text{ odd}\,,\\
M_{nn} &= -\frac{1}{3}\Big(2S(n) - 1 - (-1)^n A_n^2\Big)\,, && S(n) = \textstyle\sum_{k=0}^{n}(-1)^k A_k^2\,.
\end{aligned} \tag{2.12}$$

In the above the coefficients A and B are defined as

$$\left(\frac{1+ix}{1-ix}\right)^{1/3} = \sum_{n\,even} A_n x^n + i\sum_{n\,odd} A_n x^n\,, \quad \left(\frac{1+ix}{1-ix}\right)^{2/3} = \sum_{n\,even} B_n x^n + i\sum_{n\,odd} B_n x^n\,. \tag{2.13}$$

The first few elements of the matrix are

$$M = \begin{pmatrix} -\frac{5}{27} & 0 & \frac{32}{243\sqrt{3}} & 0 & -\frac{416\sqrt{5}}{19683} & \cdots \\ 0 & -\frac{13}{243} & 0 & \frac{512\sqrt{2}}{19683} & 0 & \cdots \\ \frac{32}{243\sqrt{3}} & 0 & -\frac{893}{19683} & 0 & \frac{1504\sqrt{5}}{59049\sqrt{3}} & \cdots \\ 0 & \frac{512\sqrt{2}}{19683} & 0 & -\frac{5125}{177147} & 0 & \cdots \\ -\frac{416\sqrt{5}}{19683} & 0 & \frac{1504\sqrt{5}}{59049\sqrt{3}} & 0 & -\frac{41165}{1594323} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{2.14}$$

The CFT method furnishes an integral expression for the elements of M^{11}. For this purpose we note the general formula[18]

$$M_{mn} = \frac{(-1)^{m+1}}{\sqrt{mn}} \oint_0 \frac{dw}{2\pi i} \oint_0 \frac{dz}{2\pi i} \frac{1}{z^m w^n} \frac{f'(z) f'(w)}{(f(z) - f(w))^2}\,, \tag{2.15}$$

where, for the three string vertex,

$$f(z) = \Big(\frac{1+iz}{1-iz}\Big)^{2/3}. \tag{2.16}$$

Both w and z integration contours are circles around the origin, with the w contour lying outside the z contour, and both contours lying inside the unit circle.

3. The $-1/3$ eigenvector of M

In this section we shall show that the matrix M has an eigenvector v^- with eigenvalue $-1/3$. This eigenvector turns out to be C odd (thus the label), and equivalently v^- satisfies $Vv^- = \frac{1}{3}v^-$, since $V = CM$. We will establish this result using the conformal field theory representation of the vertex $V_{mn} = (-1)^m M_{mn}$. The required expression was given in the previous subsection. Using integration by parts in w, and (2.16), eq. (2.15) can be turned into

$$V_{mn} = -\frac{4i}{3}\sqrt{\frac{n}{m}} \oint \frac{dw}{2\pi i} \frac{1}{w^{n+1}} \oint \frac{dz}{2\pi i} \frac{1}{z^m} \frac{1}{1+z^2} \frac{f(z)}{f(z) - f(w)}\,. \tag{3.1}$$

Numerical work suggested that the eigenvector v^- was of the form

$$v_n^- = (-1)^{(n-1)/2} \frac{1}{\sqrt{n}}\,, \quad \text{for } n \text{ odd}\,, \qquad v_n^- = 0\,, \quad \text{for } n \text{ even}\,. \tag{3.2}$$

We shall show that v^- defined in eq. (3.2) is indeed an eigenvector of V_{mn} with eigenvalue $1/3$. For regulation purposes and to understand what residues are to be picked up, take a real number a slightly bigger than one and write

$$v_n^- = \frac{a^{-n-1}}{2\sqrt{n}} \{(i)^{n-1} + (-i)^{n-1}\}\,. \tag{3.3}$$

We understand that the limit $a \to 1^+$ is to be taken. Using equations (3.1) and (3.3) we get

$$\sum_n V_{mn} v_n^- = -\frac{4i}{3}\frac{1}{\sqrt{m}} \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \frac{1}{z^m} \frac{1}{1+z^2} \frac{1}{(1+a^2w^2)} \frac{f(z)}{f(z)-f(w)} \,. \tag{3.4}$$

In order to be able to carry out the sum over n to arrive at the above equation, we must have $|w| > a^{-1}$. Thus the w integral picks up contribution from the poles at $w = \pm i/a$ and $w = z$. After this we can set $a = 1$. In this case only the $w = z$ and the $w = i$ poles contribute since $f(-i) = \infty$. Their contributions give

$$\sum_n V_{mn} v_n^- = \Big(1 - \frac{2}{3}\Big) \frac{1}{\sqrt{m}} \oint \frac{dz}{2\pi i} \frac{1}{z^m} \frac{1}{1+z^2} = \frac{1}{3} v_m^- \,. \tag{3.5}$$

This establishes the claim.

4. K_1 and its eigenvectors

In this section we shall introduce a matrix K_1 representing the action of the star-algebra derivation $K_1 = L_1 + L_{-1}$. We shall be able to find explicit forms for the eigenvectors and eigenvalues of K_1. In particular we shall see that K_1 has a non-degenerate continuous spectrum. In the next section we shall show that K_1 and M commute, and thus the eigenvectors of K_1 are eigenvectors of M. Further analysis will reveal the relation between the eigenvalues.

We begin our analysis by recalling that the operator

$$K_1 = L_1 + L_{-1} \tag{4.1}$$

is a derivation of the star algebra [19, 16]. We use its action on positively moded oscillators a_n and

$$\alpha_n \equiv \sqrt{n} a_n \tag{4.2}$$

with $n \geq 1$ to define matrices K_1 and $\widetilde{\mathrm{K}}_1$

$$\begin{aligned} [K_1, v\cdot a] &\equiv (\mathrm{K}_1 v)\cdot a - \sqrt{2} v_1 p \\ [K_1, w\cdot \alpha] &\equiv (\widetilde{\mathrm{K}}_1 w)\cdot \alpha - w_1 \alpha_0 \,. \end{aligned} \tag{4.3}$$

Here we have introduced the vector notation

$$x\cdot a \equiv \sum_{n=1}^{\infty} x_n a_n \,, \quad y\cdot\alpha \equiv \sum_{n=1}^{\infty} y_n \alpha_n \,, \tag{4.4}$$

and suppressed the Lorentz indices. Identifying $v\cdot a$ to $w\cdot\alpha$ and using eq. (4.2) we get,

$$v_n = \sqrt{n}\, w_n \,. \tag{4.5}$$

From (4.2), we have the relation

$$(\widetilde{\mathrm{K}}_1)_{mn} = \sqrt{\frac{n}{m}} (\mathrm{K}_1)_{mn} \,. \tag{4.6}$$

Using the standard commutators:

$$[L_m, \alpha_n^\mu] = -n\,\alpha_{m+n}^\mu\,, \tag{4.7}$$

we see that the matrix K_1 is symmetric, the diagonal elements are zero, and the only non-vanishing entries are one step away from the diagonal

$$\begin{aligned}(\widetilde{\mathrm{K}}_1)_{nm} &= -(n-1)\delta_{n-1,m} - (n+1)\delta_{n+1,m}\,,\\ (\mathrm{K}_1)_{nm} &= -\sqrt{n(n-1)}\delta_{n-1,m} - \sqrt{n(n+1)}\delta_{n+1,m}\,.\end{aligned} \tag{4.8}$$

Since K_1 maps twist even to twist odd states, the associated matrices anticommute with the matrix C,

$$\{\mathrm{K}_1, C\} = \{\widetilde{\mathrm{K}}_1, C\} = 0\,. \tag{4.9}$$

Finally, since K_1 is invariant under hermitian conjugation, we have:

$$[K_1, v\cdot a^\dagger] = -(\mathrm{K}_1 v)\cdot a^\dagger + \sqrt{2}v_1 p\,. \tag{4.10}$$

It is also convenient to represent vectors of type w (α_n basis) in terms of formal power series in a variable z,

$$f_w(z) \equiv \sum_{n=1}^{\infty} w_n z^n\,. \tag{4.11}$$

With this definition we note that

$$f_{Cw}(z) = f_w(-z)\,. \tag{4.12}$$

Using eq. (4.7) (or directly from eq. (4.8)) we see that the operator $K_1 = L_1 + L_{-1}$ on the basis of functions of z is represented by the differential operator:

$$\mathcal{K}_1 \equiv -(1+z^2)\frac{d}{dz}\,. \tag{4.13}$$

More specifically, with the proviso that constant terms obtained after the action of the differential operator are to be dropped, we have

$$f_{\widetilde{\mathrm{K}}_1 w}(z) = \mathcal{K}_1 f_w(z)\,. \tag{4.14}$$

It immediately follows from this equation that

$$\widetilde{\mathrm{K}}_1 w^{(\kappa)} = \kappa\, w^{(\kappa)} \quad \leftrightarrow \quad \mathcal{K}_1 f_{w^{(\kappa)}} = \kappa f_{w^{(\kappa)}} + a\,, \tag{4.15}$$

where the constant a is used to account for the fact that the action of the differential operator must be supplemented by removing the constant term. Therefore the eigenvalue problem for the infinite matrix $\widetilde{\mathrm{K}}_1$ can be studied as the eigenvalue problem for the differential operator $\mathcal{K}_1$ on the space of formal power series. The differential equation above is readily integrated to find

$$f_{w^{(\kappa)}}(z) = -\frac{1}{\kappa}\exp(-\kappa\tan^{-1}(z)) + \frac{1}{\kappa} \equiv \sum_{n=1}^{\infty} w_n^{(\kappa)} z^n\,, \tag{4.16}$$

where the overall normalization has been chosen so that $w_1^{(\kappa)} = 1$. Expanding the above in powers of z and using (4.11) one can read the coefficients $w_n^{(\kappa)}$ which, because of (4.15) provide an eigenvector of $\widetilde{\mathrm{K}}_1$. This shows that $\widetilde{\mathrm{K}}_1$ has a *non-degenerate continuous* spectrum. We can take $-\infty < \kappa < \infty$, and there is exactly one eigenvector for each value of κ. Note also that each eigenvector of $\widetilde{\mathrm{K}}_1$ provides an eigenvector $w_n^{(\kappa)}$ of K_1 with the same eigenvalue using the relation $v_n^{(\kappa)} = \sqrt{n} w_n^{(\kappa)}$. This follows from eq. (4.5).

It follows from (4.12) and (4.16) that $Cw^{(\kappa)} = -w^{(-\kappa)}$ and therefore we can form linear combinations of definite twist

$$w_{\pm}^{(\kappa)} \equiv \frac{1}{2}\Big(w^{(-\kappa)} \mp w^{(\kappa)} \Big) , \tag{4.17}$$

that satisfy

$$C\, w_{\pm}^{(\kappa)} = \pm w_{\pm}^{(\kappa)} . \tag{4.18}$$

The function representation of these eigenvectors follows from eqs. (4.17) and (4.16)

$$\begin{aligned} f_{w_-^{(\kappa)}} &= \frac{1}{\kappa} \sinh(\kappa \tan^{-1} z) , \\ f_{w_+^{(\kappa)}} &= \frac{1}{\kappa}\Big(\cosh(\kappa \tan^{-1} z) - 1 \Big) . \end{aligned} \tag{4.19}$$

These definite twist vectors are not eigenvectors of $\widetilde{\mathrm{K}}_1$ but are eigenvectors of $\widetilde{\mathrm{K}}_1^2$ with eigenvalue κ^2. For $\kappa^2 > 0$ the spectrum of $\widetilde{\mathrm{K}}_1^2$ is continuous and doubly degenerate, having, for each κ^2 a C-even and a C-odd eigenvector. For $\kappa^2 = 0$ there are also two eigenvectors. One of them arises from the $\kappa = 0$ eigenvector of $\widetilde{\mathrm{K}}_1$. This is obtained by taking the limit $\kappa \to 0$ in (4.16) and it gives the C-odd eigenvector $w^{(0)} = \tan^{-1}(z)$. Using eq. (4.5) we see that this eigenvector is precisely the $\lambda = -1/3$ eigenvector v^- of M given in (3.2). This is no coincidence, as we shall explain in the next subsection. The second zero eigenvector of $\widetilde{\mathrm{K}}_1^2$ corresponds to the function $(\tan^{-1}(z))^2$. Explicitly it takes the form

$$v_{2k}^{+} = \frac{(-1)^{k+1}}{\sqrt{2k}} \Big(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} \Big) , \quad v_{2k-1}^{+} = 0 , \quad k = 1, 2, 3, \ldots . \tag{4.20}$$

We note, however, that while the norm of v^- diverges logarithmically, the norm of v^+ has worse divergence. Furthermore v^+ is not an eigenvector of K_1. We shall also argue later that truncation of v^+ to a given level never appears as an eigenvector of the level truncated ${\mathrm{K}_1}^2$. For $\kappa^2 < 0$, or imaginary κ, the spectrum of $\widetilde{\mathrm{K}}_1^2$ is still continuous and doubly degenerate. Nevertheless, the norm of the corresponding eigenvectors seems even more divergent than the norm of the eigenvectors with real κ, and we have seen no evidence of these eigenvectors in our numerical work. This is of course consistent with the fact that K_1, being a real symmetric matrix, should only have real eigenvalues.

5. Wedge states, K_1 and M

A general wedge state $|N\rangle$ can be expressed as

$$|N\rangle = \exp\left(-\frac{1}{2} a^\dagger \cdot (CT_N) \cdot a^\dagger \right) |0\rangle \equiv \exp\Big(-E_N \Big) |0\rangle , \tag{5.1}$$

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where [20]

$$T_N = \frac{T + (-T)^{N-1}}{1 - (-T)^N}\,. \tag{5.2}$$

T has been defined in eq. (2.9). The matrices T_N are related to the Neumann coefficients of the $N-$th complete overlap string vertex, $V_N = CT_N$. Important special cases are $T_\infty = T$ (the sliver)[4] and $T_3 = M$.

The eigenvectors of T_N are the same for all the matrices in the family, and the eigenvalues are related according to the above formula. So we can simply focus on T and/or M. We shall first establish that the eigenvectors of K_1 are eigenvectors of M and of M^{12} and M^{21}. Then we find the eigenvalues of M, M^{12}, and M^{21} corresponding to a given eigenvector.

5.1 Eigenvectors of T and M

In this subsection we shall show that the eigenvectors of K_1 are eigenvectors of M and of M^{12} and M^{21}. We will do this in two stages. We first show this is true for M and all wedge state matrices T_N. Then we turn to the case of the matrices M^{12} and M^{21}.

We first note that the derivation K_1 annihilates all wedge states $|N\rangle$. Indeed, K_1 annihilates the identity, which corresponds to $N = 1$, and the SL(2,R) vacuum, which corresponds to $N = 2$. Since all higher N wedge states can be obtained by star multiplication of $N = 2$ states we have $K_1|N\rangle = 0$. We now show that as a consequence of this, the matrices T_N commute with K_1. Indeed, using eq. (5.1) we have

$$0 = K_1|N\rangle = K_1 \exp\Big(-E_N\Big)|0\rangle = -[K_1, E_N] \exp\Big(-E_N\Big)|0\rangle\,, \tag{5.3}$$

since $[K_1, E_N]$ commutes with E_N. Using (4.10), and noting that the momentum operator kills any wedge state, we have that the above equation gives

$$0 = \left(\frac{1}{2}a^\dagger \cdot (\mathrm{K}_1 C T_N + C T_N \mathrm{K}_1)\cdot a^\dagger\right)|N\rangle = \left(\frac{1}{2}a^\dagger \cdot (C\,[T_N, \mathrm{K}_1])\cdot a^\dagger\right)|N\rangle\,. \tag{5.4}$$

Since the multiplicative factor acting on the wedge states above consists of creation operators only, the factor itself must vanish identically. This implies that

$$[T_N, \mathrm{K}_1] = 0\,. \tag{5.5}$$

Since the spectrum of K_1 is non-degenerate, *all* eigenvectors of K_1 must be eigenvectors of T_N. Furthermore, since T_N commutes with C, we see from eq. (4.18) that $w^{(\pm\kappa)}$ describe degenerate eigenvectors of T_N, and $w^{(\kappa)} \pm w^{(-\kappa)}$ are simultaneous eigenvectors of T_N and C. We should note, however, that the relation $[T_N, \mathrm{K}_1] = 0$ holds only for infinite dimensional matrices T_N and K_1 and is only appproximate if we truncate T_N and K_1 to finite dimensional matrices.

[4] This follows from (5.2) if the eigenvalues of T lie in the range $[-1, 0]$, as has been found numerically. We shall return to this point later.

A similar argument can be used to show that K_1 commutes with the matrices M^{12} and M^{21}. The derivation property of K_1 implies that

$$\Big(K_1^{(1)} + K_1^{(2)} + K_1^{(3)}\Big)|V_3\rangle = 0\,, \tag{5.6}$$

where the expression for the vertex was given in (2.2) and (2.3). It suffices for the present purposes to work at zero momentum, and the above equation implies that

$$\Big(K_1^{(1)} + K_1^{(2)} + K_1^{(3)}\Big)\ \exp\Big(-\frac{1}{2}\sum_{r,s} a^{(r)\dagger} C M^{rs} a^{(s)\dagger}\Big)|0\rangle = 0\,. \tag{5.7}$$

Since the K's annihilate the vacuum we pick commutators that give

$$\Big(\frac{1}{2}\sum_{p,q} a^{(p)\dagger} C\,[\mathrm{K}_1, M^{pq}\,]\,a^{(q)\dagger}\Big)\ \exp\Big(-\frac{1}{2}\sum_{r,s} a^{(r)\dagger} C M^{rs} a^{(s)\dagger}\Big)|0\rangle = 0\,. \tag{5.8}$$

This condition implies that $[K_1, M^{pq}] = 0$, as claimed. Once again, the non-degeneracy of the K_1 spectrum implies that K_1 eigenvectors are eigenvectors of M^{12} and M^{21}.

5.2 Relating the eigenvalues of M and K_1 via the B matrices

While the K_1 operator helps us determine the eigenvectors of M and T_N, so far it has not given us information about the corresponding eigenvalues as the precise relation between M and K_1 has not been found. In this subsection we shall find this relation. We do this by introducing a new matrix B, much simpler than M, and that shares all the eigenvectors of M. The relation of B to T (or M) is calculable analytically and thus the relation between their eigenvalues is fixed. Furthermore the action of B on the eigenvector $w_n^{(\kappa)}$ will also be calculable analytically. This in turn, will determine the action of M and T on $w_n^{(\kappa)}$.

We define B as the leading expansion of T_N, when N is very close to two:

$$T_{2+\epsilon} = \epsilon\, B + O(\epsilon^2)\,. \tag{5.9}$$

Expanding the formula (5.2) we find the relation of B with the sliver matrix T,

$$B = -\frac{T\ln(-T)}{1-T^2}\,. \tag{5.10}$$

Notice that as $T \to -1$, $B \to -1/2$, and as $T \to 0$, $B \to 0$. So the spectrum of B is expected to lie on the interval $[-1/2, 0]$.

To obtain an expression for the matrix elements of B we consider in more detail the wedge state $|2+\epsilon\rangle$. We have, on the one hand,

$$|2+\epsilon\rangle = \exp(\epsilon V_-)|0\rangle = |0\rangle + \epsilon V_-|0\rangle + O(\epsilon^2) \tag{5.11}$$

for an appropriate vector field

$$V_- = \sum_{n=2}^{\infty} v_n L_{-n}\,. \tag{5.12}$$

To find V_-, recall that wedge states $|N\rangle = \exp\left(\sum_{k=2}^{\infty} c_k^{(N)} L_{-k}\right)|0\rangle$ are defined by requiring [16]

$$\exp\left(\sum_{k=2}^{\infty} c_k^{(N)} z^{k+1}\partial_z\right) z = \frac{N}{2}\tan\left(\frac{2}{N}\arctan(z)\right). \tag{5.13}$$

From eqs. (5.11)–(5.13) we have $c_k^{(2+\epsilon)} = \epsilon\, v_k$. Expanding the right hand side of (5.13) for $N = 2+\epsilon$ we get

$$\sum_n v_n z^{n+1} = \frac{1}{2}(z - (1+z^2)\tan^{-1} z)\,. \tag{5.14}$$

This gives[5]

$$V_- = \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n-1)(2n+1)} L_{-2n}\,. \tag{5.15}$$

On the other hand we also have from (5.1) and (5.9) that

$$|2+\epsilon\rangle = \exp\left(-\epsilon\frac{1}{2}a^\dagger\cdot(CB)\cdot a^\dagger\right)|0\rangle = |0\rangle - \epsilon\,\frac{1}{2}a^\dagger\cdot(CB)\cdot a^\dagger|0\rangle\; + O(\epsilon^2)\,. \tag{5.16}$$

Comparing the right hand sides of (5.11) and (5.16), using eq. (5.15), and the equation

$$L_{-m} = \frac{1}{2}\sum_p \alpha^\mu_{-m+p}\alpha^\nu_{-p}\eta_{\mu\nu} \quad \text{for} \quad m > 0\,, \tag{5.17}$$

we finally find

$$\begin{aligned} B_{mn} &\equiv -\frac{(-1)^{\frac{n-m}{2}}\sqrt{m\,n}}{(m+n)^2-1} \quad \text{for } n+m \text{ even} \\ B_{mn} &\equiv 0 \quad \text{for } n+m \text{ odd}\,. \end{aligned} \tag{5.18}$$

Note that the matrix B is much simpler than the matrix T or M.

Since T_N commutes with K_1 for every N, so must B. Thus the eigenvectors $v_n^{(\kappa)} = \sqrt{n}w_n^{(\kappa)}$ must also be eigenvectors of B. Our goal now is to find an expression for the eigenvalues $\beta(\kappa)$ of B associated to the eigenvectors $v^{(\kappa)}$. For this we consider the eigenvalue equation

$$\sum_{n\geq 1} B_{mn} v_n^{(\kappa)} = \beta(\kappa) v_m^{(\kappa)}\,. \tag{5.19}$$

Since this relation holds for every m we have, in particular,

$$\beta(\kappa) = \frac{1}{v_1^{(\kappa)}}\sum_{n\geq 1} B_{1n} v_n^{(\kappa)} = \frac{1}{w_1^{(\kappa)}}\sum_{q=1}^{\infty}\frac{(-1)^q}{2q+1}\,w^{(\kappa)}_{2q-1}\,. \tag{5.20}$$

If we define

$$F(z) = \sum_{q=1}^{\infty}\frac{(-1)^q}{2q+1}\; w^{(\kappa)}_{2q-1}\, z^{2q+1}\,, \tag{5.21}$$

[5]This vector field has also been considered independently by Schnabl[24].

then we may rewrite (5.20) as

$$\beta(\kappa) = \frac{F(1)}{w_1^{(\kappa)}} = F(1)\,, \tag{5.22}$$

since, as seen from eq. (4.16), $w_1^{(\kappa)} = 1$. On the other hand, we have,

$$\frac{dF(z)}{dz} = \sum_{q=1}^{\infty} (-1)^q w_{2q-1}^{(\kappa)} z^{2q} = \frac{1}{2}\, iz \left(f_{w^{(\kappa)}}(iz) - f_{w^{(\kappa)}}(-iz) \right), \tag{5.23}$$

where $f_{w^{(\kappa)}}(z)$ has been defined in eq. (4.16). We can easily integrate this equation (with the boundary condition $F(0) = 0$) to get

$$\beta(\kappa) = F(1) = -\frac{1}{\kappa} \int_0^1 dz\, z \sin(\kappa \tanh^{-1}(z)) = -\frac{1}{2} \frac{\kappa\pi/2}{\sinh(\kappa\pi/2)}\,. \tag{5.24}$$

This is the eigenvalue of the matrix B associated to the eigenvector $v^{(\kappa)}$. Using eq. (5.10) we can determine the corresponding eigenvalue of T to be:

$$\tau(\kappa) = -e^{-|\kappa|\pi/2}\,. \tag{5.25}$$

In deriving (5.25) we have noted that eq. (5.10) does not determine T uniquely for a given A, and we need some additional input. This comes from the requirement that the eigenvalue of T lies between -1 and 0, a fact found in numerical experiments.

Finally, using eq. (5.2), we can determine the eigenvalue $\mu(\kappa)$ of $M = T_3$ to be

$$\mu(\kappa) = \frac{\tau(\kappa) + (\tau(\kappa))^2}{1 + (\tau(\kappa))^3} = -\frac{1}{1 + 2\cosh(\kappa\pi/2)}\,. \tag{5.26}$$

This completes the determination of the eigenvectors and eigenvalues of the matrix M. Furthermore, eq. (5.26) also provides us a simple expression for M in terms of the matrix K_1:

$$M = -(1 + 2\cosh(\mathrm{K}_1\pi/2))^{-1}\,. \tag{5.27}$$

5.3 Diagonalization of ρ_1, ρ_2, M^{12} and M^{21}

In section 2 we defined the two real symmetric projectors ρ_1 and ρ_2 (eqn. (2.8)). Having shown that the eigenvectors of K_1 are eigenvectors of T and M^{rs}, it follows that they must also be eigenvectors of ρ_1 and ρ_2. This implies that the simultaneous eigenstates of ρ_1, ρ_2 and M are given by the vectors $v_n^{(\kappa)} = \sqrt{n}\, w_n^{(\kappa)}$, with $w_n^{(\kappa)}$ defined as in (4.16). On the other hand ρ_i do not commute with C, instead a conjugation by C converts ρ_1 to $\rho_2 = 1 - \rho_1$ and vice versa. Thus the 0 and 1 eigenvalues of ρ_1 (or ρ_2) must get exchanged under the action of C.

One could now ask: what is the eigenvalue $\lambda_i(\kappa)$ of $\rho_i(\kappa)$ for a given value of κ? $\lambda_i(\kappa)$ must take values 0 or 1. Furthermore we have the relation

$$\lambda_1(\kappa) + \lambda_2(\kappa) = 1\,. \tag{5.28}$$

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Also from the twist properties of $\lambda_i(\kappa)$, we have

$$\lambda_1(-\kappa) = \lambda_2(\kappa) = 1 - \lambda_1(\kappa)\,. \tag{5.29}$$

By continuity in κ, the only possible choices seem to be that $\lambda_1(\kappa)$, for example, be equal to one for all positive κ, or equal to one for all negative κ. Numerical results show that the first possibility is realised:

$$\lambda_1(\kappa) = \begin{cases} 1 & \text{for} \quad \kappa > 0\,, \\ 0 & \text{for} \quad \kappa < 0\,. \end{cases} \tag{5.30}$$

Thus ρ_1 and ρ_2 project onto eigenstates of K_1 with positive and negative eigenvalues respectively.

Note that acting on an eigenvector of M with precisely $-1/3$ eigenvalue, ρ_1 and ρ_2 become ill-defined since $(1+T)$ vanishes acting on such a state. It is natural to define ρ_i such that both ρ_1 and ρ_2 annihilate this state. Since ρ_1 and ρ_2 project onto the modes of the right- and the left-half of the string respectively [17, 21, 22, 23], we can interpret the states with $\kappa > 0$, $\kappa < 0$ and $\kappa = 0$ as the modes of the right half-string, left half-string and the string mid-point respectively.

Using eqs. (2.7) and (5.26) one readily shows that

$$\mu^{12}(\kappa) - \mu^{21}(\kappa) = \pm\sqrt{(1-\mu(\kappa))(1+3\mu(\kappa))} = \pm\frac{2\sinh(\pi\kappa/2)}{1+2\cosh(\pi\kappa/2)}\,, \tag{5.31}$$

where $\mu^{12}(\kappa)$ and $\mu^{21}(\kappa)$ are the eigenvalues of M^{12} and M^{21} for the eigenvector $w^{(\kappa)}$. Together with the relation $\mu^{12}(\kappa) + \mu^{21}(\kappa) = 1 - \mu(\kappa)$ following from the first equation in (2.7), the values of $\mu^{12}(\kappa)$ and $\mu^{21}(\kappa)$ would be determined in terms of κ were it not for the square root sign ambiguity above. This ambiguity can be resolved using (5.30). Indeed, consider $\kappa > 0$ in which case $\tau(\kappa) = -\exp(-\kappa\pi/2)$ (eqn. (5.25)) and $\rho_2 w^{(\kappa)} = 0$. Using the explicit form of ρ_2 in (2.8) one can check that $\rho_2 w^{(\kappa)} = 0$ requires choosing the top sign in (5.31). We thus conclude that $v^{(\kappa)}$ is an eigenstate of M^{12} and M^{21} with eigenvalues

$$\begin{aligned} \mu^{12}(\kappa) &= \frac{1+\cosh(\pi\kappa/2)+\sinh(\pi\kappa/2)}{1+2\cosh(\pi\kappa/2)}\,, \\ \mu^{21}(\kappa) &= \frac{1+\cosh(\pi\kappa/2)-\sinh(\pi\kappa/2)}{1+2\cosh(\pi\kappa/2)}\,. \end{aligned} \tag{5.32}$$

5.4 String functionals and the spectrum

In this subsection we give the functional interpretation of the eigenvalue equations we have considered so far. In this setup one writes the eigenvalue equations as functional constraints satisfied by the string functionals associated to wedge states or the sliver. This represents a generalization of the considerations of Moore and Taylor [14] who interpreted the C-odd $\kappa = 0$ eigenvector of the sliver as the existence of a flat direction in the sliver functional.

To develop this approach in general we define

$$T_N\, v_\pm^{(\kappa)} = \mu_N(\kappa) v_\pm^{(\kappa)}\,, \qquad C v_\pm^{(\kappa)} = \pm v_\pm^{(\kappa)}\,, \tag{5.33}$$

where T_N is the general wedge state matrix appearing in (5.1) and (5.2), the vectors $v_\pm^{(\kappa)}$ are defined in eqs. (1.2), (1.4), and $\mu_N(\kappa)$ is the eigenvalue, calculable from (5.2) and (5.25). It follows from (5.33) and (5.1) that

$$v_\pm^{(\kappa)} \cdot \Big(a \pm \mu_N(\kappa)\, a^\dagger \Big) |N\rangle = 0\,, \tag{5.34}$$

or equivalently

$$v_\pm^{(\kappa)} \cdot \Big((1 \pm \mu_N(\kappa))(a + a^\dagger) + (1 \mp \mu_N(\kappa))\,(a - a^\dagger) \Big) |N\rangle = 0\,. \tag{5.35}$$

The translation to functional language is effected with the relations[6]

$$\hat{x}_n = \frac{i}{\sqrt{n}}(a_n - a_n^\dagger)\,, \qquad \hat{p}_n = -i\frac{\partial}{\partial x_n} = \frac{\sqrt{n}}{2}(a_n + a_n^\dagger)\,, \tag{5.36}$$

which allow us to rewrite (5.35) as

$$\sum_{n\geq 1} \Big\{ 2(1 \pm \mu_N(\kappa))(w_\pm^{(\kappa)})_n \frac{\partial}{\partial x_n} + (1 \mp \mu_N(\kappa))\, n (w_\pm^{(\kappa)})_n\, x_n \Big\} \langle X(\sigma)|N\rangle = 0\,, \tag{5.37}$$

where we used the standard properties $\langle X(\sigma)|\hat{x}_n = \langle X(\sigma)|x_n$ and $\langle X(\sigma)|\hat{p}_n = -i\frac{\partial}{\partial x_n}\langle X(\sigma)|$ of the position eigenstate. Making use of the mode expansions

$$X(\sigma) = x_0 + \sqrt{2}\sum_{n=1}^{\infty} x_n\, \cos n\sigma\,, \qquad \pi\,\frac{\delta}{\delta X(\sigma)} = \frac{\partial}{\partial x_0} + \sqrt{2}\sum_{n=1}^{\infty}\frac{\partial}{\partial x_n}\,\cos n\sigma\,, \tag{5.38}$$

we can rewrite the constraint in (5.37)

$$\Big\{ 2(1 \pm \mu_N(\kappa)) \int_0^\pi d\sigma\, F_\pm^{(\kappa)}(\sigma)\pi\,\frac{\delta}{\delta X(\sigma)} + (1 \mp \mu_N(\kappa)) \int_0^\pi d\sigma \widetilde{F}_\pm^{(\kappa)}(\sigma)\, X(\sigma) \Big\} \langle X(\sigma)|N\rangle = 0\,, \tag{5.39}$$

where the functions F and $\widetilde{F}$ are simply related to the formal functions $f_{w_\pm^{(\kappa)}}(z)$ (see (4.19)) representing the eigenvectors

$$\begin{aligned} F_\pm^{(\kappa)}(\sigma) &= \sum_{n\geq 1} (w_\pm^{(\kappa)})_n\, \cos n\sigma \,=\, \mathrm{Re}\Big\{ f_{w_\pm^{(\kappa)}}(z)\Big|_{z=e^{i\sigma}} \Big\}\,, \\ \widetilde{F}_\pm^{(\kappa)}(\sigma) &= \sum_{n\geq 1} n(w_\pm^{(\kappa)})_n\, \cos n\sigma = \mathrm{Re}\Big\{ z\frac{d}{dz} f_{w_\pm^{(\kappa)}}(z)\Big|_{z=e^{i\sigma}} \Big\}\,. \end{aligned} \tag{5.40}$$

Equation (5.39), with the definitions in (5.40), is the functional constraint associated to the eigenvalue equation (5.33). For the particular case of the sliver ($N \to \infty$) we have that T has a C-odd eigenvector with eigenvalue $\mu = -1$ associated to $\kappa = 0$. Thus if we take the lower sign in (5.39) the second term vanishes. Given this, the constraint simply reduces to the existence of a flat direction on the sliver functional. This flat direction is defined by the invariance of the functional under the variation $X(\sigma) \to X(\sigma) + \epsilon F_-^{(0)}(\sigma)$. Here

$$F_-^{(0)}(\sigma) = \sum_{n\geq 1} (w_-^{(0)})_n\, \cos n\sigma = \mathrm{Re}\Big(\tan^{-1}(e^{i\sigma}) \Big) = \frac{\pi}{4}\begin{cases} +1, & 0 \leq \sigma \leq \pi/2 \\ -1, & \pi/2 \leq \sigma \leq \pi\,, \end{cases} \tag{5.41}$$

[6]These are compatible with the normalization convention given in footnote 3.

which is the conclusion of [14] that the sliver functional is invariant under opposite rigid displacements of the left and right halves of the string. Indeed whenever the second term in (5.39) vanishes we have a flat direction $X(\sigma) \to X(\sigma) + \epsilon F_-^{(\kappa)}(\sigma)$. Whenever the first term vanishes, the functional $\langle X(\sigma)|N\rangle$ must contain a delta function of the form $\delta(h(X(\sigma))$ where $h(X(\sigma)) = \int_0^\pi d\sigma \widetilde{F}_\pm^{(\kappa)}(\sigma)\, X(\sigma)$. An example of this case was provided in [14]: a C-even eigenvector of eigenvalue one that is present for the instantonic sliver, in which case the delta function constraint requires the midpoint of the string to lie at the instanton location.

Given the distribution of eigenvalues for the wedge states $|N\rangle$, which lie on the interval $[-1 + (2/N), 0)$, the eigenvalue (-1) can only be attained for the sliver. Thus, for all these cases, and for the sliver eigenvalues different from -1, the constraint is more general. It is a functional differential constraint on the wave function.

6. Spectral density and finite level analysis

In the previous sections we have constructed the exact eigenstates and eigenvectors of the matrices K_1, B, T and M. The question that we shall address in this section is: how do these results get modified if we work with the truncation of these matrices to square $L \times L$ matrices ? Clearly, with finite size matrices the eigenvalues will form a discrete set. We explain how this quantization of the continuous spectrum arises. We then describe our numerical results.

6.1 Quantization condition on the eigenvalues for finite matrices

We begin by analyzing the matrix K_1 truncated to a square $L \times L$ matrix. By a small abuse in language we simply call this the level L truncation of K_1, and we denote it by K_{1L}. Since $(\mathrm{K}_1)_{mn}$ vanishes unless $n = m \pm 1$, given an eigenstate $v_n^{(\kappa)}$ of the infinite dimensional matrix K_1, the restriction $\bar{v}^{(\kappa)}$ of $v^{(\kappa)}$ to level L, defined as the L dimensional vector:

$$\bar{v}_n^{(\kappa)} = v_n^{(\kappa)} \quad \text{for } 1 \le n \le L\,, \tag{6.1}$$

will be an exact eigenstate of K_{1L} if

$$v_{L+1}^{(\kappa)} = 0\,. \tag{6.2}$$

Since $v_n^{(\kappa)} = \sqrt{n} w_n^{(\kappa)}$ with $w_n^{(\kappa)}$ defined through the expansion of eq. (4.16), we see that the eigenvalues κ of K_{1L} are determined by the equation:

$$\kappa^{-1} \left(\partial_z^{L+1} \exp(-\kappa \tan^{-1} z)\right)\Big|_{z=0} = 0\,. \tag{6.3}$$

The left hand side of eq. (6.3) gives a polynomial in κ of degree L, and the L solutions are the eigenvalues of the level L truncation of K_1. Since K_{1L} is a real symmetric matrix, all its eigenvalues are guaranteed to be real. For odd L, the polynomial is odd under $\kappa \to -\kappa$ and hence always has a solution $\kappa = 0$. The corresponding eigenvector is given by the restriction of v_n^- defined in eq. (3.2) to first L entries. On the other hand for even L the left hand side of eq. (6.3) is an even polynomial in κ. The constant term in this polynomial, proportional to $(\partial_z^{L+1}\tan^{-1}z)|_{z=0}$, is non-zero for every L, and hence there are no zero roots of this polynomial. Thus for even L we do not have any eigenvector of K_{1L} with zero eigenvalue.

Having explained how the discrete spectrum of the truncated K_1 matrix arises, we now turn to the eigenvalues of the matrices B, T and M. Let R be any one of the infinite dimensional matrices B, T or M discussed earlier, and $u_n(\rho)$ denote its exact eigenvector with eigenvalue ρ:

$$R_{mn}\,u_n(\rho) = \rho\,u_m(\rho)\,. \tag{6.4}$$

It will be convenient to take u to be a simultaneous eigenvector of R and C (and hence of ${\mathrm{K}_1}^2$), given by $u_n = v^{(\kappa)}_{\pm n}$, with $v^{(\kappa)}_{\pm}$ defined as in eqs. (1.2) and (1.4). Furthermore, let R_L be the restriction of R to level L, and $\bar{u}(\rho)$ be the restriction of the vector $u(\rho)$ to level $n \le L$, defined in a manner analogous to eq. (6.1). Then

$$\sum_{n=1}^{L} R_{Lmn}\,\bar{u}_n(\rho) = \rho\,\bar{u}_m(\rho) - \sum_{n>L} R_{mn}\,u_n(\rho)\,. \tag{6.5}$$

Now suppose for large n, and fixed m, the leading contribution to R_{mn} has the form:

$$R_{mn} \simeq f(m)g(n)\,, \quad n \gg m\,, \tag{6.6}$$

for some functions f and g. In that case $\bar{u}$ will be an eigenvector of R_L with eigenvalue ρ to leading order if

$$\sum_{n>L} g(n)\,u_n(\rho) = 0\,. \tag{6.7}$$

This equation is the approximate quantization condition for the eigenvalues. The specific values of ρ_i for which (6.7) is satisfied make the last term in (6.5) vanish to leading order and therefore $(\rho_i, \bar{u}(\rho_i))$ are approximate eigenvalues and eigenvectors of R_L. Thus the eigenvector associated to a discrete eigenvalue is simply the level L truncation of the exact eigenvector of R associated to that eigenvalue.

As already stated above, we choose the vectors u_n to be simultaneous eigenvectors of R and C (and hence of ${\mathrm{K}_1}^2$). The exact eigenstates of these infinite dimensional matrices are given in terms of the expansion coefficients of the functions shown in (4.19). The discrete set of κ's satisfying eq. (6.6) then gives us the approximate eigenstates of R_L. The corresponding eigenvalues are computed by evaluating $\rho(\kappa)$ given in eqs. (5.24), (5.25) and (5.26) for $R = B$, T and M respectively. Since the expressions for the function $g(n)$ introduced in (6.6) in general would differ for B, T and M, at any level L the sets of quantized values of κ need not agree for these different matrices. As a result the corresponding eigenvectors will also differ from each other, reflecting the fact that the relations between the matrices K_1, B, T and M, which hold at infinite level, no longer hold for the level truncated matrices.

To see examples of the factorization property (6.6), we note that if R corresponds to the matrix B defined in eq. (5.18), then for a C-odd eigenvector we need only consider m

and n odd, and we find

$$f(m) = (-1)^{\frac{m+1}{2}}\sqrt{m}, \qquad g(n) = (-1)^{\frac{n-1}{2}} n^{-3/2}\,. \tag{6.8}$$

On the other hand, for a C-even eigenvector we need only consider m and n even, and

$$f(m) = -(-1)^{\frac{m}{2}}\sqrt{m}, \qquad g(n) = (-1)^{\frac{n}{2}} n^{-3/2}\,. \tag{6.9}$$

The factorization described in (6.6) also holds for M as can be seen, for example, using equation (4.32) of [14]. In the case of m and n odd, relevant for C-odd eigenvectors, one finds $g(n) = (-1)^{\frac{n-1}{2}} n^{-\frac{7}{6}}$, and for m and n even, relevant for C-even eigenvectors, one finds $g(n) = (-1)^{\frac{n}{2}} n^{-\frac{7}{6}}$.

6.2 Eigenvalue distribution function

Given the continuous spectrum of eigenvalues, one could study the density of eigenvalues. Since the analysis is simplest for the eigenvalues of K_1, we could first find the density $\rho^L_{\mathrm{K}_1}(\kappa)$ of eigenvalues of ${\mathrm{K}_1}_L$, where $\int_{\kappa_0}^{\kappa_1} \rho^L_{\mathrm{K}_1}(\kappa) d\kappa$ would give the number of eigenvalues of ${\mathrm{K}_1}_L$ lying between κ_0 and κ_1. We can then take the $L \to \infty$ limit to compute the asymptotic density $\rho_{\mathrm{K}_1}(\kappa)$. This can then be used to compute the density of eigenvalues of another matrix (say M) via the relation:

$$\rho_M(\mu) = 2\left(\frac{d\mu}{d\kappa}\right)^{-1} \rho_{\mathrm{K}_1}(\kappa)\,. \tag{6.10}$$

The factor of two in the above formula comes from the fact that two different values of κ, differing by a sign, give the same μ.

We shall find it more convenient to compute the eigenvalue densities of C-odd and C-even eigenvectors of ${\mathrm{K}_1}^2_L$ separately, and then combine the results to find the eigenvalue density of ${\mathrm{K}_1}_L$. The eigenvalue equation for the matrix $\widetilde{\mathrm{K}}_1^2$:

$$\widetilde{\mathrm{K}}_1^2\, w_{(\kappa^2)} = \kappa^2\, w_{(\kappa^2)}\,, \tag{6.11}$$

leads to the recursion relation (we drop the eigenvalue subscript for simplicity)

$$w_{n+2} = \frac{w_n(\kappa^2 - 2n^2) - (n-1)(n-2)w_{n-2}}{(n+1)(n+2)}\,. \tag{6.12}$$

Let us just take $n = 2k$ and introduce

$$t_k \equiv w_{2k}\,, \tag{6.13}$$

to consider the C-even case. The equation becomes:

$$2(2k+1)(k+1)t_{k+1} = -t_k(8k^2 - \kappa^2) - 2(2k-1)(k-1)t_{k-1}\,. \tag{6.14}$$

In order to get an eigenvector of $\widetilde{{\mathrm{K}_1}^2_L}$, — a matrix whose nonvanishing entries only extend two steps from the diagonal, — we need, in analogy to (6.2), the condition:

$$t_{[L/2]+1} = 0\,, \tag{6.15}$$

where $[L/2]$ denotes the integral part of $L/2$. Clearly, to solve this equation for large L, we need to find the behaviour of t_k for large k. This is the problem we shall now address. We note in passing that eq. (6.15) is never satisfied by the C-even, $\kappa^2 = 0$ eigenvector v^+ of ${\rm K_1}^2$ given in eq. (4.20). Thus this eigenvector never generates an eigenvector of the level truncated ${\rm K_1}^2$.

Let us introduce s_k as

$$s_k \equiv k\, t_k\, (-1)^k\,. \tag{6.16}$$

eq. (6.14) now becomes

$$(2k+1)s_{k+1} + (2k-1)\, s_{k-1} = s_k(4k - \frac{\kappa^2}{2k})\,. \tag{6.17}$$

We look for slowly varying solutions of this equation so that we can regard s_k as a continuous function $s(k)$ that does not vary much when k changes by one. We can then turn this into a differential equation

$$s'' + \frac{1}{k}s' + \frac{\kappa^2}{4}\frac{1}{k^2}s = 0\,, \tag{6.18}$$

where the leading neglected term in the left hand side is $\frac{1}{6}\frac{s'''}{k}$. Let us ignore this term for now — this will be justified later. Then the differential equation is solved by:

$$s(k) \sim k^\alpha\,, \qquad \alpha^2 = -\frac{\kappa^2}{4}\,. \tag{6.19}$$

Therefore, for positive κ^2 we get

$$s \sim A(\kappa)\cos\left(\frac{|\kappa|}{2}\ln k\right) + B(\kappa)\sin\left(\frac{|\kappa|}{2}\ln k\right). \tag{6.20}$$

With $s \sim k^\alpha$ the neglected term in the differential equation

$$\frac{1}{6}\frac{s'''}{k} \sim \frac{1}{6}\alpha(\alpha-1)(\alpha-2)\frac{s}{k^4} \sim \frac{1}{6}(\alpha-2)\frac{s''}{k^2} \ll s''\,, \tag{6.21}$$

is therefore much smaller than any other term, for any finite α whenever k is large. The same happens for terms with even more derivatives. Thus this solution can be trusted. Going back we have

$$\begin{aligned} w_{2k} &\sim \frac{(-1)^k}{k}\left[A(\kappa)\cos\left(\frac{|\kappa|}{2}\ln k\right) + B(\kappa)\sin\left(\frac{|\kappa|}{2}\ln k\right)\right] \\ &\sim \frac{(-1)^k}{k}C(\kappa)\sin\Big(\frac{|\kappa|}{2}\ln k + \phi(\kappa)\Big)\,, \end{aligned} \tag{6.22}$$

for some constants C and ϕ. (This behaviour is also seen in explicit computations.)

Substituting eq. (6.22) into (6.15) we get

$$\frac{1}{2}|\kappa|\, \ln([L/2]+1) + \phi(\kappa) = n\pi\,, \tag{6.23}$$

for some integer n. The difference between successive values of $|\kappa|$ satisfying eq. (6.23), for $\ln L \gg |\phi'(\kappa)|$, is:

$$\Delta|\kappa| = \frac{2\pi}{\ln L}\,. \tag{6.24}$$

The analysis of the odd eigenvectors gives a similar equation. Thus we have two states in the interval of $\Delta|\kappa| = 2\pi/\ln L$. On the other hand, since the $\widetilde{\mathrm{K}}_1$ (and K_1) eigenvalues come in pairs with opposite sign, we see that if we study the eigenvalues κ of the matrix K_1, they have a uniform spacing given by $2\pi/\ln L$ for both positive and negative κ. This gives

$$\rho^L_{\mathrm{K}_1}(\kappa) = \frac{\ln L}{2\pi} \,. \tag{6.25}$$

Of course this is valid only for finite values of κ. Since the total number of eigenvalues for finite L is finite, clearly the distribution gets modified for large $|\kappa|$.

The eigenvalue density for the large but finite level L truncation of M now follows by the standard transformation given in (6.10) with the derivative evaluated using (5.26). The result is the one quoted in (1.6).

As a consistency check, we now confirm the uniform density of eigenvalues in κ space by an analysis of M. The method is similar to the one used for K_1 and uses eqs. (6.7), (6.22), and the asymptotic behaviour of $g(n)$ ($\sim (-1)^{n/2} n^{-7/6}$) discussed below eq. (6.9). We find the following relation for the C-even eigenvector of M:

$$\sum_{k=[L/2]+1}^{\infty} k^{-5/3} \sin\left(\frac{|\kappa|}{2}\ln k + \phi(\kappa)\right) = 0 \,. \tag{6.26}$$

Since the summand is a slowly varying function of k for large k, we can replace the sum by an integral over k, and after performing the integral, get an equation of the form:

$$\frac{1}{2}|\kappa| \ln([L/2]+1) + \chi(\kappa) = n\pi \,, \tag{6.27}$$

where $\chi(\kappa)$ is another phase factor. Thus for large L, the eigenvalues of M are uniformly distributed in the κ space, with the same density as given in (6.25), as expected. Note, however, that since $\chi(\kappa)$ is different from $\phi(\kappa)$, the precise values of κ appearing in the solution of the eigenvalue equation for M differ from that appearing in the eigenvalue equation for K_1.

6.3 Numerical tests of the spectrum

In this subsection we shall present some numerical experiments we have done on the calculation of eigenvalues and eigenvectors of ${\mathrm{K}_1}_L$ and M_L at finite level L. This work confirms the theoretical expectations developed in the previous sections.

Numerical analysis of the eigenvalues and the eigenvectors of the matrix M is carried out by constructing these matrices following the general procedure given in [2] and reviewed in section 2, and then finding the eigenvalues and eigenvectors of the truncated matrix M_L at level L. It turns out that even working at finite level, where the largest eigenvalue is still about 10% away from the value $-1/3$, the eigenvectors follow very accurately the prediction from the ${\mathrm{K}_1}^2$ eigenvectors as given in eq. (4.19). Given an eigenvector of M and the corresponding eigenvalue determined numerically, we determine κ using eq. (5.26) and then use this to predict the eigenvector using eq. (4.19). For example, calculating

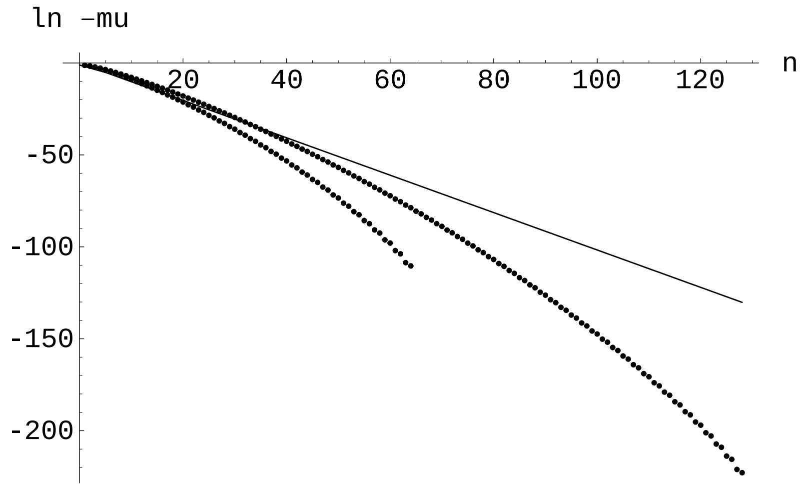

Figure 1: This figure shows two plots of eigenvalues μ_n of the level truncated matrix M_L. In one of them $L = 64$, and in the other $L = 128$. On the horizontal axis we have n referring to the n-th eigenvalue with $n = 1$ labelling the smallest eigenvalue (closest to $-1/3$). On the vertical axis we show $\ln(-\mu_n)$. Note that the eigenvalues become small very fast. The solid line shows the predicted curve for $L = 128$, ignoring corrections of order $1/\ln L$.

the spectrum of M to level $L = 2048$ we find that the eigenvalue of largest magnitude is $\mu_0 = -0.310141$, and it corresponds to the C-odd eigenvector whose first components are

$$w_1 = 1, \quad w_3 = -0.318455, \quad w_5 = 0.185188, \quad w_7 = -0.129081\,, \quad w_9 = 0.098372\,. \tag{6.28}$$

Note that just like the eigenvalue μ_0 is far from the limiting value $-1/3$, the eigenvector is also reasonably far from the vector $w_-^{(0)}$ associated with the expansion of $\tan^{-1} z$. Making use of (5.26) we can find the associated value $\kappa(\mu_0) = \pm 0.298782$. Using (4.19) the C-odd eigenvector of $\widetilde{\mathrm{K}}_1^2$ is

$$f_\kappa^{(-)}(z) = \frac{1}{\kappa} \sinh(\kappa \tanh^{-1} z)\,. \tag{6.29}$$

Expanded in powers of z, for $\kappa(\mu_0)$ given above, this gives

$$f_{\kappa(\mu_0)}^{(-)}(z) = z - 0.318455\ z^3 + 0.185188z^5 - 0.129081z^7 + 0.098372z^9 + \cdots \tag{6.30}$$

in remarkable agreement with (6.28). The explanation for this phenomenon has already been discussed in the paragraph below eq. (6.9).

The spectrum of eigenvalues of M computed using level truncation show some regular pattern. Let μ_n denote the n-th eigenvalue of M arranged in ascending order so that $n = 1$ corresponds to the eigenvalue closest to $-1/3$. The eigenvalues corresponding to C-even and C-odd eigenvectors alternate as their magnitude decrease monotonically. In fact, the eigenvalues go to zero very rapidly and thus one must work with many digits in order to obtain accurate results. The eigenvalue spectrum is illustrated in figure. 1, where $\ln(-\mu_n)$ is plotted against n for the case of level truncations of M with $L = 64$ and $L = 128$. The

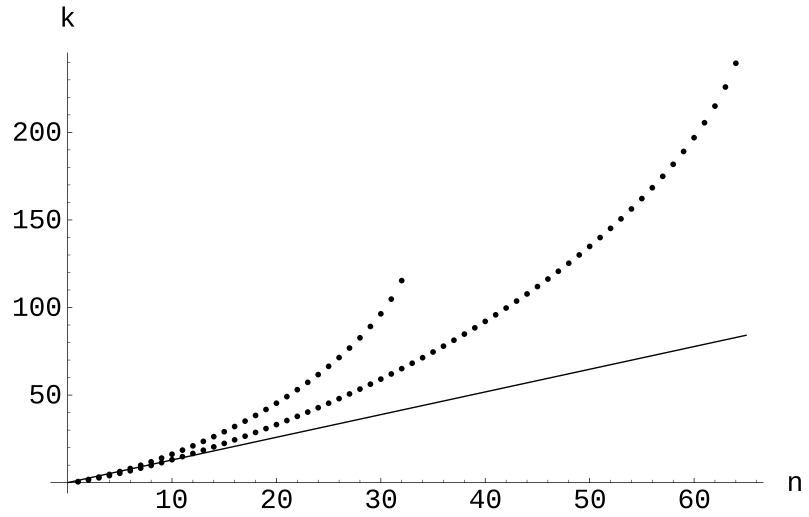

Figure 2: This figure shows two plots of the positive eigenvalues κ_n of the level truncated matrix K_1. In one of them $L = 64$, and in the other $L = 128$. On the horizontal axis we have n referring to the n-th eigenvalue with $n = 1$ labelling the smallest eigenvalue. On the vertical axis we show κ_n. The solid line shows the predicted answer for $L = 128$ ignoring corrections of order $1/\ln L$.

value of n for any given μ measures the quantity $\int_{-1/3}^{\mu} \rho_M^L(\mu')d\mu'$. As we can see on the tail ends of the distributions, the pairing of C-even and C-odd eigenvectors emerges as pairs of dots are seen to coalesce. Moreover as we can see, the curve of the level 128 eigenvalues lies above the curve of the level 64 eigenvalues. This is consistent with the emergence of a continuous spectrum in the $L \to \infty$ level, a fact proven in the previous sections.

We can compute the predicted answer for $\rho_M^L(\mu)$ using eqs. (6.10) and (6.25). This gives

$$n(\mu) = \int_{-1/3}^{\mu} \rho_M^L(\mu')d\mu' = 2\int_0^{\kappa(\mu)} \rho_{\mathrm{K}_1}^L(\kappa')d\kappa' = \frac{\ln L}{\pi}\,\kappa(\mu)\,, \tag{6.31}$$

in the $L \to \infty$ limit. Using eq. (5.26) we can rewrite this as:

$$n(\mu) = 2\,\frac{\ln L}{\pi^2}\ln\left\{\frac{\mu + 1 + \sqrt{(1+3\mu)(1-\mu)}}{2|\mu|}\right\}. \tag{6.32}$$

This predicted curve, computed for $L = 128$, has been shown by the solid line in figure 1.

Numerical evaluation of the eigenvalues of K_1 is straightforward using eq. (6.3). Figure 2 shows the plot of $\kappa(n)$ vs. n, where $\kappa(n)$ denotes the n-th positive eigenvalue of κ, with $n = 1$ representing the lowest positive (or zero) eigenvalue. This data can be regarded as a plot of $\int_0^{\kappa} \rho_{\mathrm{K}_1}^L(\kappa')d\kappa'$, — the value of n for any given κ gives the number of K_1 eigenvalues in the range $[0, \kappa]$. According to eq. (6.25), the predicted answer for this quantity is:

$$n(\kappa) = \int_0^{\kappa} \rho_{\mathrm{K}_1}^L(\kappa')d\kappa' = \frac{\ln L}{2\pi}\kappa\,. \tag{6.33}$$

We show by the solid line the predicted curve for $L = 128$. For small κ this matches reasonably well the numerical results.

7. Open questions

In this paper we have diagonalized the matrices required for star multiplication of zero-momentum string functionals. One immediate extension that should be contemplated is that of finding the analogous results for the matrices M' introduced in [2, 6, 7] that include entries for zero modes. These results would also determine the spectrum of the matrix defining the instantonic sliver. Indeed, complete knowledge of the spectral distributions may allow us to calculate analytically the ratio

$$\frac{\det(1-M')^{3/4}\det(1+3M')^{1/4}}{\det(1-M)^{3/4}\det(1+3M)^{1/4}} \tag{7.1}$$

which enters into the evaluation of the ratios of tensions of D-branes differing by one dimension ([7, eqn. (3.10)],). Currently the computation of this ratio can only be done by level expansion, though a BCFT argument can be used to show that the ratio of tensions must arise correctly [25].

Another problem of importance is the diagonalization of the Neumann coefficients which appear in the computation of the star product in the ghost sector. This problem, however, is automatically solved given our results, and those in ref. [3, eqs.(4.6), (4.7) and (4.11)] relating the matter and the ghost Neumann coefficients. In particular, it follows from [3] that the diagonal Neumann matrix $\widetilde{M}^{11} = \widetilde{C}\widetilde{V}^{11}$ appearing in the computation of the star product in the Siegel gauge is related to the matrix M analyzed here via the relation:

$$\widetilde{M}^{11} = -M(1+2M)^{-1}\,, \tag{7.2}$$

up to a simiarity transformation involving scaling of c_{-n} and b_{-n} by $\sqrt{n}$ and $1/\sqrt{n}$ respectively. Thus the eigenvalues and eigenvectors of $\widetilde{M}^{11}$ are determined in terms of those of M.

In discussing the spectrum of infinite matrices in this paper we have not restricted ourselves to eigenvectors with finite norm. Indeed, under the obvious norm $\sum_n |v_n^{(\kappa)}|^2 = \infty$. Thus our eigenvectors are not vectors in a Hilbert space. This does not seem to be a problem. The eigenvectors satisfy the eigenvalue equations in a clear sense: the sums involved converge. Moreover, our eigenvectors are the ones that indeed appear to emerge in the finite level analysis. Given the success of level expansion, this should be taken as strong evidence that we need to deal with this kind of eigenvectors. Indeed, our work may help understand the proper normalization condition that should be imposed on the eigenvectors.

A rigorous mathematical description of the spectrum of M would go beyond the present work. The spectrum of linear operators is in general larger than the set of eigenvalues (the so-called point spectrum). Even our eigenvalues and eigenvectors are not on a rigorous footing since we have not attempted to define concretely the vector space where M acts. Despite these reservations our computations do seem to capture physically relevant aspects of the properties of M. In particular our results are consistent with the point of view that the spectrum of M is obtained as the infinite level limit of the finite $L \times L$ truncation of M.

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At a more basic level, the analysis in this paper should help build the experience that will allow a proper understanding of the set of string fields for which the star algebra is defined. Currently open string field theory can be viewed as a formulation of string theory with non-perturbative information, for example, the existence of D-branes. If the proper space of string field is clearly defined, this would turn open string field theory into a *non-perturbative definition* of string theory.

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